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Sphere of influence graphs and the L_∞-metric

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Abstract

We introduce sphere of influence graphs (SIGs) in the L_∞-metric and study their elementary properties. We argue that SIGs defined with the L_∞-metric are superior to Euclidean SIGs of Toussaint in capturing low-level perceptual information in certain dot patterns. Every graph without isolated vertices is a SIG in the L_∞-metric for all sufficiently high dimensions, and this allows us to define a graphical parameter, the SIG-dimension, that is akin to boxicity. We determine the SIG-dimensions for some classes of graphs and obtain inequalities for others.

Keywords: Sphere of influence graph; Proximity graph; Intersection graph

1. Introduction

Let $M=(\mathbf{M},\rho)$ be a metric space with point set $\mathbf{M}$ and metric ρ, and let $\mathscr{X}=\{X_1,\ldots,X_n\}$ be a set of n points ($n\geqslant 2$) in M. Let

$$r_i=\min\{\rho(X_i,X_j)\colon j\neq i\}\quad (i=1,\ldots,n)$$

denote the minimum distance between X_i and any other point in $\mathscr{X}$. The open ball

$$B_i=\{X\in\mathbf{M}:\rho(X_i,X)<r_i\} \tag{1}$$

with center X_i and radius r_i is the *sphere of influence* at X_i ($i=1,\ldots,n$). The *sphere of influence graph* $\mathrm{SIG}(M,\mathscr{X})$ has vertex set $\mathscr{X}$ with edges corresponding to pairs of intersecting spheres of influence; thus, the edge set is

$$\{[X_i,X_j]\colon B_i\cap B_j\neq\emptyset,\ i\neq j\}.$$

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The graph G is an *abstract sphere of influence graph* in M (or simply an M-SIG) provided G is isomorphic to SIG($M,\mathscr{X}$) for some subset $\mathscr{X}$ of M. The set $\mathscr{X}$ *realizes* the graph G in M.

Sphere of influence graphs (SIGs) are simultaneously intersection graphs and proximity graphs. Note that an induced subgraph of an M-SIG need not be an M-SIG [3,7]. This non-hereditary property complicates the problem of characterizing sphere of influence graphs.

Toussaint [11–13] introduced sphere of influence graphs with M as the Euclidean plane in order to model situations in pattern recognition and computer vision. We refer to Toussaint's graphs as *Euclidean planar* sphere of influence graphs. Very little progress has been made toward characterizing the class of Euclidean planar SIGs despite the efforts of a number of researchers. (See the survey [7].) For instance, the conjecture [3] that the complete graph $K(9)$ is not a Euclidean planar SIG remains unresolved. Some information about Euclidean planar SIGs has been obtained by Michael and Quint [6,8] by studying SIGs that arise from general metric spaces as defined above; the results in [8] reveal that many of the properties of SIGs depend only on the triangle inequality and not on deeper properties of the underlying metric space.

In this paper we initiate the study of SIGs that arise from the metric space $M_\infty^d = (\mathbf{R}^d, \rho_\infty)$ whose metric ρ_∞ is induced by the L_∞-norm on the real vector space $\mathbf{R}^d$. Thus, the distance between $X=(x_1,\dots,x_d)$ and $Y=(y_1,\dots,y_d)$ is

$$\rho_\infty(X,Y)=\max\{|x_i-y_i|\colon i=1,\dots,d\}.$$

Each "sphere" of influence in M_∞^d is a *d-cube*, i.e., a hypercube in $\mathbf{R}^d$ whose edges are parallel to the coordinate axes. The simple geometry of the spheres of influence allows us to obtain stronger results about M_∞^d-SIGs than for SIGs in other metric spaces.

2. Dot patterns and the L_∞-metric

In his seminal paper Toussaint [11] provided numerous examples of dot patterns in the plane (e.g., random patterns, block lettering, mazes, optical illusions) together with the corresponding Euclidean planar sphere of influence graphs. These examples suggest that Euclidean planar SIGs capture low-level perceptual information present in dot patterns better than the more familiar types of proximity graphs. In this section we argue that for special types of dot patterns the SIGs associated with the L_∞-metric are superior to the Euclidean planar SIGs.

Fig. 1(a) is typical of the dot patterns displayed on electronic signs in which an illuminated subset of lattice points spells out a message. Such a set will more likely contain configurations of points that are collinear and arranged diagonally. The L_∞-metric will assign points in these configurations a radius equal to 1, while the Euclidean metric assigns the larger radius $2^{1/2}$. With the L_∞-metric we thus expect fewer "extraneous" edges in the sphere of influence graph, and a more satisfactory sphere of influence graph. Indeed, Fig. 1(b) shows the Euclidean SIG for the dot pattern, while Fig. 1(c) uses the L_∞-metric. Fig. 1(d) is the SIG using the L_1-metric, another plausible

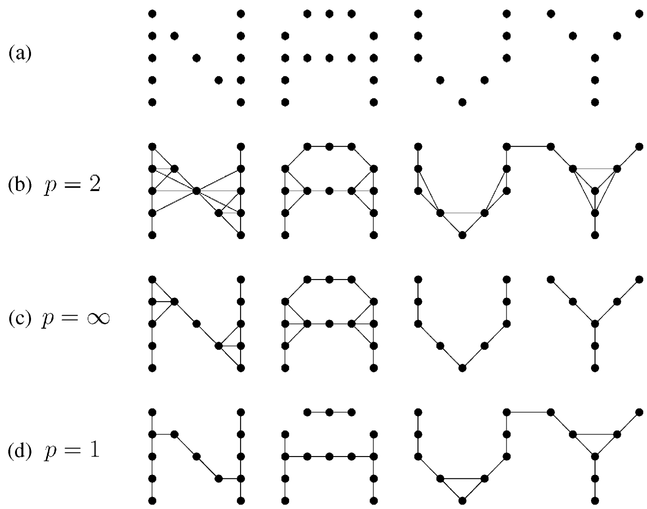

Fig. 1. Comparison of SIGs under different metrics. (a) A dot pattern in the plane. The axes are oriented parallel to the edges of the page. (b)–(d) The corresponding SIGs in the L_p-metric for $p=2$ (Euclidean planar SIG), $p=\infty$, and $p=1$.

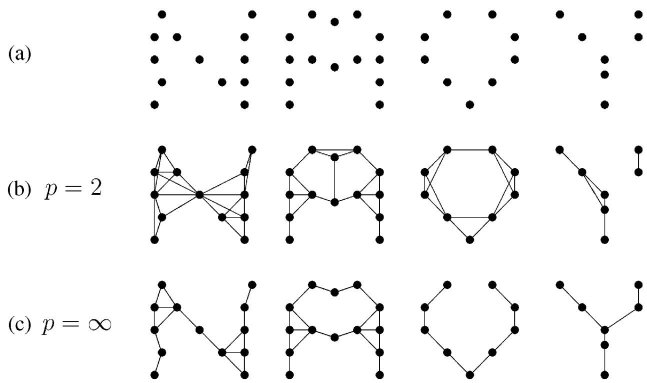

Fig. 2. (a) A perturbation of the dot pattern of Fig. 1. (b) the Euclidean planar SIG. (c) the SIG under the L_∞-metric.

candidate. To our eyes the L_∞-metric provides the most pleasing graph for this dot pattern; examples with dot patterns for other alphanumeric characters support this opinion.

The SIGs under the L_∞-metric are also more robust (compared to the Euclidean planar SIGs) for our dot patterns, as can be seen from a comparison of Figs. 1 and 2. Here is one reason for this robustness. Consider a configuration of collinear points arranged either horizontally or vertically. (Such configurations occur frequently in the electronic sign context.) Both the Euclidean metric and the L_∞-metric give rise to paths in the corresponding sphere of influence graphs, as desired. However, suppose one randomly perturbs the points slightly. (Such perturbations could arise from errors in the measurements taken by a robot's sensory apparatus.) Then the lack of collinearity among the perturbed points will give rise to a subgraph $K(3)$ in the Euclidean planar SIG. On the other hand, the SIG under the L_∞-metric will still be a path.

We remark that although L_∞-metric does seem to capture certain patterns better than the Euclidean metric, it possesses one liability for pattern recognition; the L_∞-metric is not rotationally invariant.

3. The SIG-dimension of a graph

Recall that an *interval graph* is the intersection graph of a family of intervals on the real line. The intervals need not be distinct, and both open and closed intervals are allowed. A *d-box* is the Cartesian product of d intervals, i.e., a parallelotope in $\mathbf{R}^d$ whose edges are parallel to the coordinate axes. A *d-box graph* is the intersection graph of a family of d-boxes. If all the d-boxes are required to have edge length 1, then we refer to a *d-cube graph.* Roberts [9] defined the *boxicity* and *cubicity* of the graph G by

$$\mathrm{box}(G) = \min\{d : G \text{ is a } d\text{-box graph}\},$$
$$\mathrm{cub}(G) = \min\{d : G \text{ is a } d\text{-cube graph}\}.$$

(If G is a complete graph, then $\mathrm{box}(G)=\mathrm{cub}(G)=0$ by convention.) Thus $\mathrm{box}(G)=1$ if and only if G is a non-complete interval graph.

We now introduce the analogous parameter for SIGs Let $G=(V,E)$ be a graph without isolated vertices. The *SIG-dimension* of G is

$$\mathrm{sig}(G) = \min\{d : G \text{ is an } M_\infty^d\text{-SIG}\}.$$

The need to exclude graphs with isolated vertices is clear. It is easy to see that an M_∞^d-SIG is also an M_∞^e-SIG for all $e \geqslant d$.

Theorem 1. *Let G be a graph with n vertices, none of which is isolated. Then* $\mathrm{box}(G) \leqslant \mathrm{sig}(G) \leqslant n-1$.

Proof. Every M_∞^d-SIG is a d-box graph and the lower bound for the SIG-dimension follows. To prove the upper bound, delete column n from the matrix $A+2I$, where A denotes an adjacency matrix of G, and I is the identity matrix of order n. Let X_i denote the ith row of the resulting n by $n-1$ matrix ($i=1,\ldots,n$), and view the row vector X_i as a point in $\mathbf{R}^{n-1}$. Then every sphere of influence for the set $\mathscr{X}=\{X_1,\ldots,X_n\}$ has radius 1. Also, adjacent vertices of G correspond to points at distance 1 from one another in the L_∞-metric, while non-adjacent vertices correspond to points at distance 2 from one another. Therefore $\mathscr{X}$ realizes G as an M_∞^{n-1}-SIG. $\square$

Theorem 1 implies that the SIG-dimension is well-defined for graphs without isolated vertices. In the next several sections of this paper we seek formulas and bounds for the SIG-dimensions of special families of graphs in terms of more familiar parameters. Let us make two elementary observations now. First, *if G is a disconnected* *graph whose connected components* $G_1,\ldots,G_m$ *all have at least two vertices, then* $\mathrm{sig}(G)=\max\{\mathrm{sig}(G_i),\ i=1,\ldots,m\}$. Thus in our study of SIG-dimensions we may restrict attention to connected graphs. Second, $\mathrm{sig}(G)=1$ *if and only if each connected component of G is a non-trivial path.* Characterizing graphs with fixed SIG-dimension $d \geqslant 2$ is a much more difficult problem. (See Section 9.)

4. Complete graphs

With any two points $X=(x_1,\ldots,x_d)$ and $Y=(y_1,\ldots,y_d)$ in $\mathbf{R}^d$ we associate the *metric color*

$$\min\{k\colon |x_k-y_k| \geqslant |x_j-y_j| \text{ for all } j=1,\ldots,d\}.$$

Thus the metric color records the smallest index in $\{1,\ldots,d\}$ that defines the distance between X and Y under the L_∞-metric. The *metric coloring* of a set $\mathscr{X}$ of points in $\mathbf{R}^d$ associates a metric color with each pair of points in $\mathscr{X}$. A hyperplane in $\mathbf{R}^d$ is *standard* of *color* k provided it is orthogonal to the kth coordinate axis.

The following theorem has a Ramsey flavor.

Theorem 2. *The edges of an* M_∞^d-SIG *may be colored with d colors so that every complete subgraph on n vertices uses at least* $\lceil \log_2(n)\rceil$ *colors.*

Proof. Let the set $\mathscr{X}$ realize the graph G as an M_∞^d-SIG, and let $\mathscr{X}'$ denote a subset of $\mathscr{X}$ corresponding to a complete subgraph $K(n)$. Assume that $X_1,\ldots,X_t$ are the points (in order) of $\mathscr{X}'$ that correspond to a monochromatic odd cycle of color k in the metric coloring of $\mathscr{X}'$. Then the cyclic sequence $x_k^{(1)},\ldots,x_k^{(t)}$ of kth components contains a monotonic subsequence of three consecutive terms, say, $x_k^{(1)}<x_k^{(2)}<x_k^{(3)}$. Now the spheres of influence B_1 and B_3 are separated by the standard hyperplane of color k through X_2, which implies that X_1 and X_3 are not adjacent in G, a contradiction. Thus there is no monochromatic odd cycle in the metric coloring of $\mathscr{X}'$. Suppose that the metric coloring of $\mathscr{X}'$ uses λ colors. Then we have shown that $K(n)$ is decomposed into λ bipartite graphs. However, the λ associated bipartitions of $\mathscr{X}'$ will fail to distinguish between (and associate a metric color with) some pair of vertices of $K(n)$ if $n>2^\lambda$. Therefore $\lambda \geqslant \lceil \log_2(n)\rceil$. □

Corollary 3. *Let G be a graph with no isolated vertices. If* $K(n)$ *is a subgraph of* G, *then* $\mathrm{sig}(G)\geqslant\lceil\log_2(n)\rceil$. *Also*, $\mathrm{sig}(K(n))=\lceil\log_2(n)\rceil$.

Proof. The first implication follows from Theorem 2. If $n\leqslant 2^d$, then $K(n)$ is realized by a set of n points in $\mathbf{R}^d$ with all components in $\{\pm 1\}$; every sphere of influence has radius 2 and contains the origin. Therefore $\mathrm{sig}(K(n))=\lceil\log_2(n)\rceil$. □

We remark that a proof that $\mathrm{sig}(K(n))=\lceil\log_2(n)\rceil$ based on Helly's Theorem may be extracted from the discussion following Conjecture 4.2 in [2].

5. Complete multipartite graphs

In this section we obtain lower and upper bounds for the SIG-dimension of a complete q-partite graph ($q \geqslant 2$) that are similar to Roberts' exact formula [9]

$$\mathrm{cub}(K(n_1,\ldots,n_q)) = \sum_{i=1}^{q} \lceil \log_2(n_i) \rceil$$

for the cubicity. In fact, we obtain bounds for the SIG-dimension of graphs containing $K(n_1,\ldots,n_q)$ as a certain type of subgraph. We say that a subgraph G' of the graph G is *omnipresent* provided every edge of G occurs in an induced subgraph isomorphic to G'. We say that the subgraph of $G=(V,E)$ induced by the vertex subset V' is *prominent* provided each vertex not in V' is adjacent to every vertex in V' or to no vertex in V'. For instance, every graph is both an omnipresent and a prominent subgraph of itself.

Theorem 4. *Let G be a graph with no isolated vertices. Suppose that G has an induced complete q-partite graph $K(n_1,\ldots,n_q)$ that is either omnipresent or prominent ($q \geqslant 2$). Then*

$$\mathrm{sig}(G) \geqslant \max\left\{ \lceil \log_2(q) \rceil, \sum_{i=1}^{q} \lceil \log_2(n_i) \rceil \right\}. \tag{2}$$

Proof. Because $K(q)$ is a subgraph of $K(n_1,\ldots,n_q)$, Corollary 3 implies that $\mathrm{sig}(K(n_1,\ldots,n_q) \geqslant \lceil \log_2(q) \rceil$.

Let the set $\mathscr{X}$ realize G as an M_∞^d-SIG, and let $\mathscr{X}' = \mathscr{X}_1 \cup \cdots \cup \mathscr{X}_q$ be a subset of $\mathscr{X}$ corresponding to an induced subgraph $K(n_1,\ldots,n_q)$, where $\mathscr{X}_i$ contains the vertices of the ith partite set ($i=1,\ldots,q$). Thus $|\mathscr{X}_i| = n_i$, and the corresponding spheres of influence in the set $\mathscr{B}_i$ are disjoint ($i=1,\ldots,q$). Let $\mathscr{B}' = \mathscr{B}_1 \cup \cdots \cup \mathscr{B}_q$ denote the set of spheres of influence in $\mathrm{SIG}(M_\infty^d, \mathscr{X})$ corresponding to the points in $\mathscr{X}'$. We shall exhibit a set $\mathscr{H}$ of mutually orthogonal hyperplanes in $\mathbf{R}^d$ with $|\mathscr{H}| = \sum_{i=1}^q \lceil \log_2(n_i) \rceil$, which will imply $d \geqslant \sum_{i=1}^q \lceil \log_2(n_i) \rceil$, and so prove (2). If the metric coloring of $\mathscr{X}_i$ uses metric color k, then there is a standard hyperplane of color k that separates two spheres of influence in $\mathscr{B}_i$ ($i=1,\ldots,q$). Choose one such hyperplane for each color used in $\mathscr{X}_i$, and let $\mathscr{H}_i$ denote the resulting set of mutually orthogonal hyperplanes. If $i \neq j$, then every hyperplane in $\mathscr{H}_i$ must be orthogonal to every hyperplane in $\mathscr{H}_j$. For the only alternative is that some two hyperplanes are parallel, which is impossible since every sphere of influence in $\mathscr{B}_i$ must intersect every hyperplane in $\mathscr{H}_j$ in order for $\mathscr{X}'$ to correspond to an induced complete multipartite subgraph of G.

To complete the proof it suffices to show that $|\mathscr{H}_i| \geq \lceil \log_2(n_i) \rceil$, for then $\mathscr{H} = \mathscr{H}_1 \cup \cdots \cup \mathscr{H}_q$ is our desired set of orthogonal hyperplanes in $\mathbf{R}^d$. Assume that $|\mathscr{H}_i| < \lceil \log_2(n_i) \rceil$. Then the metric coloring of $\mathscr{X}_i$ uses fewer than $\lceil \log_2(n_i) \rceil$ colors, and, as in the proof of Theorem 2, there are three points X_1, X_2, and X_3 in a monochromatic odd cycle of color k in $\mathscr{X}_i$ whose kth components satisfy $x_k^{(1)} < x_k^{(2)} < x_k^{(3)}$. The spheres of influence B_1, B_2, and B_3 are pairwise disjoint, and thus there is a standard hyperplane H_{12} (resp., H_{23}) of color k that separates B_1 and B_2 (resp., B_2 and B_3).

The ball B_2 lies between H_{12} and H_{23}, and hence the distance between these parallel hyperplanes (and the distance between B_1 and B_3) is at least twice the radius of B_2. Let Y denote the point in $\mathscr{X} - \{X_2\}$ nearest to X_2. If $K(n_1,\dots,n_q)$ is an omnipresent subgraph of G, then we select $\mathscr{X}'$ so that $\{X_2, Y\} \subseteq \mathscr{X}'$. Now whether $K(n_1,\dots,n_q)$ is omnipresent or prominent, the sphere of influence at Y must intersect both B_1 and B_3 in order for $\mathscr{X}'$ to correspond to $K(n_1,\dots,n_q)$. However, this is impossible because the sphere of influence at Y is no larger than B_2. □

Corollary 5. *Let G be a bipartite graph whose partite sets have degree sequences $1 \leqslant d_1 \leqslant \cdots \leqslant d_m$ and $1 \leqslant e_1 \leqslant \cdots \leqslant e_n$. Then* $\mathrm{sig}(G) \geqslant \lceil \log_2(\max\{d_1, e_1\}) \rceil$.

Proof. The graphs $K(1, d_1)$ and $K(1, e_1)$ are omnipresent subgraphs of G, and the result follows from Theorem 4. □

A graph is *triangle-free* provided it contains no subgraph isomorphic to $K(3)$. For each non-negative integer p define

$$\tilde{p} = \begin{cases} 1 & \text{if } p = 0, \\ p & \text{if } p \geqslant 1. \end{cases}$$

Corollary 6. *Let G be a triangle-free graph with degree sequence $1 = d_1 = \cdots = d_p < d_{p+1} \leqslant \cdots \leqslant d_n$, where $0 \leqslant p \leqslant n-1$. Then* $\mathrm{sig}(G) \geqslant \lceil \log_2(d_{\tilde{p}+1}) \rceil$.

Proof. Without loss of generality no connected component of G is isomorphic to $K(2)$ since deletion of such a component does not alter $\mathrm{sig}(G)$. Now $K(1, d_{\tilde{p}+1})$ is an omnipresent subgraph of G, and the result follows from Theorem 4. □

We now give a construction that gives an upper bound for the SIG-dimension of a complete multipartite graph.

Theorem 7. *Let $n_1,\dots,n_q$ be q positive integers $(q \geqslant 2)$, exactly p of which equal 1. Then*

$$\mathrm{sig}(K(n_1,\dots,n_q)) \leqslant \lceil \log_2(\tilde{p}) \rceil + \sum_{i=1}^{q} \lceil \log_2(n_i) \rceil. \tag{3}$$

If $p \leqslant 1$, then

$$\mathrm{sig}(K(n_1,\dots,n_q)) = \sum_{i=1}^{q} \lceil \log_2(n_i) \rceil. \tag{4}$$

Proof. Without loss of generality $1 = n_1 = \cdots = n_p < n_{p+1} \leqslant \cdots \leqslant n_q$, where $0 \leqslant p \leqslant q$. If $p = q$, then the result follows from Corollary 3. Suppose that $p < q$. For each positive integer m we let $A(m)$ (resp., $A'(m)$) denote any matrix of 1's and -1's (resp., 0's and 1's) of size m by $\lceil \log_2(m) \rceil$ with distinct rows. If $p = 0$, define the direct sum $A = A(n_1) \oplus \cdots \oplus A(n_q)$. If $p = 1$, construct the matrix A by prepending a row of 0's to $A(n_2) \oplus \cdots \oplus A(n_q)$. If $p \geqslant 2$, let $A = A'(p) \oplus A(n_{p+1}) \oplus \cdots \oplus A(n_q)$. In each case the matrix A has $d = \lceil \log_2(\tilde{p}) \rceil + \sum_{i=1}^{q} \lceil \log_2(n_i) \rceil$ columns, and the set of row vectors of A realizes $K(n_1, \ldots, n_q)$ as an M_∞^d-SIG; each sphere of influence has radius 1, and non-adjacent vertices correspond to points at distance 2. This establishes (3). The lower and upper bounds in (2) and (3) agree when $p \leqslant 1$, and formula (4) follows. □

6. Isometries, the SIG-dimension, and trees

A connected graph $G = (V, E)$ may be viewed as a metric space (V, ρ) on the vertex set V with metric defined by the usual distance function ρ in G that counts the number of edges in a shortest path between two vertices. An *isometry* from G to the metric space $M_\infty^d = (\mathbf{R}^d, \rho_\infty)$ is a function $f : V \to \mathbf{R}^d$ that preserves distance, i.e., for all vertices x and y

$$\rho(x, y) = \rho_\infty(f(x), f(y)).$$

In their work on isometries from finite metric spaces to normed linear spaces Linial et al. [5] discuss the graphical parameter

$$\overline{\dim}(G) = \min\{d : \text{there exists an isometry from } G \text{ to } M_\infty^d\},$$

which we now relate to the SIG-dimension.

Theorem 8. *Let G be a connected graph. Then* $\operatorname{sig}(G) \leqslant \overline{\dim}(G)$.

Proof. Let f be an isometry from $G = (V, E)$ to M_∞^d. Then the set $\mathscr{X} = \{f(x): x \in V\}$ realizes G as an M_∞^d-SIG since every sphere of influence has radius 1, and two vertices are non-adjacent in G if and only if their images under f are at distance at least 2 in $\mathbf{R}^d$. □

Theorem 8 yields an upper bound for the SIG-dimension of a tree.

Theorem 9. *Let T be a tree with degree sequence $1 = d_1 = \cdots = d_p < d_{p+1} \leqslant \cdots \leqslant d_n$ $(n \geqslant 3)$. Then* $\lceil \log_2(d_{p+1}) \rceil \leqslant \operatorname{sig}(T) \leqslant \lfloor 1.71 \log_2(p) \rfloor$.

Proof. Corollary 6 gives the lower bound. The proof of Theorem 5.3 in [5] shows that a tree T with p vertices of degree 1 satisfies $\overline{\dim}(T) \leqslant C \log_2(p)$, where $C = (\log_2(3) - 1)^{-1} = 1.7095\ldots$. Apply Theorem 8 to prove the upper bound. □

Although an upper bound for $\overline{\dim}(G)$ translates to an upper bound for $\operatorname{sig}(G)$, the ratio $\overline{\dim}(G)/\operatorname{sig}(G)$ can be arbitrarily large. For instance, the results in Section 5 of [5] imply that $\overline{\dim}(C_n) \geqslant (n-5)/4$ for cycles of length n, whereas it is not difficult to show that $\operatorname{sig}(C_n) = 2$.

7. The unitary SIG-dimension

In this section we introduce a graphical parameter $\mathrm{sig}^*(G)$ that bounds $\mathrm{sig}(G)$ from above (Proposition 10(a)); is well-behaved with respect to induced subgraphs (Theorem 11); and may be computed by a finite algorithm (see the discussion following Problem 24).

Let $G=(V,E)$ be an M_∞^d-SIG with vertex set $V=\{v_1,\dots,v_n\}$ and a corresponding point set $\mathscr{X}=\{X_1,\dots,X_n\}$ in $\mathbf{R}^d$. Then G is a *unitary sphere of influence graph* in M_∞^d (or a unitary M_∞^d-SIG) provided

$$\rho(v_i,v_j)=1 \quad \text{in } G \quad \text{implies} \quad \rho_\infty(X_i,X_j)=1 \quad \text{in } M_\infty^d.$$

In particular, every sphere of influence in $\mathrm{SIG}(M_\infty^d,\mathscr{X})$ has radius 1. Moreover, in a unitary M_∞^d-SIG

$$\rho(v_i,v_j)=s \ \text{ iff } \ \rho_\infty(X_i,X_j)=s \quad (s=1,2). \tag{5}$$

Thus every isometric embedding of a graph into M_∞^d yields a unitary M_∞^d-SIG. However, in a unitary M_∞^d-SIG implication (5) may fail for vertices at distance $s\geqslant 3$.

For a graph G without isolated vertices we define the *unitary* SIG-*dimension* by

$$\mathrm{sig}^*(G)=\min\{d : G \text{ is a unitary } M_\infty^d\text{-SIG}\}.$$

The relationships in the following proposition are clear from our preceding observations.

Proposition 10. *If G is a graph without isolated vertices, then*
(a) $\mathrm{cub}(G)\leqslant \mathrm{sig}^*(G)$;
(b) $\mathrm{sig}(G)\leqslant \mathrm{sig}^*(G)\leqslant \overline{\dim}(G)$;
(c) $\mathrm{sig}^*(G)=\overline{\dim}(G)$ *if the diameter of G is* <3.

Theorem 11. *Let G' be an induced subgraph of G, and suppose that neither G nor G' has isolated vertices. Then* $\mathrm{sig}^*(G')\leqslant \mathrm{sig}^*(G)$.

Proof. Let $\mathscr{X}$ realize G as a unitary M_∞^d-SIG, and let $\mathscr{X}'$ be the subset of $\mathscr{X}$ corresponding to the vertices in G'. Because G' is an induced subgraph with no isolated vertices, $\mathscr{X}'$ realizes G' as a unitary M_∞^d-SIG, and the inequality follows. □

Lemma 12. *Let G be a unitary M_∞^d-SIG. Then G is realized as a unitary M_∞^d-SIG by a point set with all integer components.*

Proof. Let $\mathscr{X}=\{X_1,\dots,X_n\}$ realize G as a unitary M_∞^d-SIG, where $X_i=(x_1^{(i)},\dots,x_d^{(i)})$. Define the point $\lfloor X_i\rfloor=(\lfloor x_1^{(i)}\rfloor,\dots,\lfloor x_d^{(i)}\rfloor)$ for $i=1,\dots,n$. Then

$$\lfloor\mathscr{X}\rfloor=\{\lfloor X_1\rfloor,\dots,\lfloor X_d\rfloor\}$$

is a set of points in $\mathbf{R}^d$ with all components integers. We claim that $\lfloor\mathscr{X}\rfloor$ realizes G as a unitary M_∞^d-SIG. Suppose that $\rho_\infty(X_i,X_j)=1$. Then $|x_k^{(i)}-x_k^{(j)}|\leqslant 1$ for all k, and equality holds for at least one k. Thus $\rho_\infty(\lfloor X_i\rfloor,\lfloor X_j\rfloor)=1$. It follows that every sphere of influence in $\mathrm{SIG}(M_\infty^d,\lfloor\mathscr{X}\rfloor)$ has radius 1. Now suppose that $\rho_\infty(X_i,X_j)\geqslant 2$. Then

$|x_k^{(i)} - x_k^{(j)}| \geqslant 2$ for some k, and thus $\rho_\infty(\lfloor X_i \rfloor, \lfloor X_j \rfloor) \geqslant 2$. Thus $\lfloor \mathscr{X} \rfloor$ realizes G as a unitary M_∞^d-SIG. □

Let Z_∞^d denote the infinite M_∞^d-SIG corresponding to the integer lattice points in $\mathbf{R}^d$. Thus the vertices of Z_∞^d are the integer lattice points in $\mathbf{R}^d$, and vertices X and Y are adjacent provided $\rho_\infty(X, Y) = 1$. The next result shows that Z_∞^d plays a fundamental role in the study of the unitary SIG-dimension.

Theorem 13. *The unitary* SIG-*dimension of a graph G without isolated vertices is given by*

$$\mathrm{sig}^*(G) = \min\{d : G \text{ is an induced subgraph of } Z_\infty^d\}.$$

Proof. Let $\mathscr{X}$ realize G as a unitary M_∞^d-SIG, where $d = \mathrm{sig}^*(G)$. By Lemma 12 we may assume that each point in $\mathscr{X}$ is an integer lattice point in $\mathbf{R}^d$. It follows that G is an induced subgraph of Z_∞^d. Also, by Theorem 11 every finite induced subgraph of Z_∞^d without isolated vertices is a unitary M_∞^d-SIG. □

Corollary 14. *The maximum degree of a vertex in a unitary M_∞^d-*SIG *is at most $3^d - 1$.*

Proof. Every vertex in Z_∞^d has degree $3^d - 1$, and the result follows from Theorem 13. □

Corollary 14 implies that the ratio $\mathrm{sig}^*(G)/\mathrm{sig}(G)$ can be arbitrarily large. Let $K^0(1,n)$ be the graph obtained by sub-dividing each edge of $K(1,n)$. Then $\mathrm{sig}^*(K^0(1,n)) \geqslant \lceil \log_3(n+1) \rceil$ by Corollary 14, while $\mathrm{sig}(K^0(1,n)) = 2$. Also, note that we have an example of the non-monotonicity of the SIG-dimension with respect to taking induced subgraphs; the graph $K^0(1,n)$ has smaller SIG-dimension than its induced subgraph $K(1,n)$ since $2 = \mathrm{sig}(K^0(1,n)) < \mathrm{sig}(K(1,n)) = \lceil \log_2(n) \rceil$ for $n \geqslant 5$.

8. Closed SIGs

The edges of a *closed sphere of influence graph* in the metric space M are defined by the intersections of the closed balls

$$\bar{B}_i = \{X \in \mathbf{M} : \rho(X_i, X) \leqslant r_i\}$$

instead of the open balls in (1). We refer to these graphs as M-CSIGs. The notation and terminology for M-CSIGs are similar to those for M-SIGs. In this section we discuss results about M_∞^d-CSIGs that are analogous to some of our earlier results about M_∞^d-SIGs. We first recall a well-known extension of Helly's Theorem: *If a family of closed d-boxes intersect pairwise, then some point in $\mathbf{R}^d$ is in all of the d-boxes.*

Theorem 15. *The complete graph $K(3^d)$ is an M_∞^d-*CSIG, *but the complete graph $K(3^d + 1)$ is not an M_∞^d-*CSIG.

Proof. The set of 3^d points in $\mathbf{R}^d$ with components in $\{0, \pm 1\}$ realizes a complete graph on 3^d vertices. Suppose that $\mathscr{X} = \{X_1, \ldots, X_n\}$ realizes $K(n)$ as an M_∞^d-CSIG, where n is maximal. Then the corresponding closed spheres of influence intersect pairwise. By Helly's Theorem there is a point in all of the spheres of influence. Without loss of generality this common point is the origin O. Note that $O \in \mathscr{X}$; otherwise $\mathscr{X} \cup \{O\}$ realizes $K(n+1)$, contrary to the maximality of n. The *sign sequence* of a vector $(x_1, \ldots, x_d)$ in $\mathbf{R}^d$ is $\sigma(x_1), \ldots, \sigma(x_d)$, where $\sigma(x)$ records the sign of x as $0, +1$, or -1. Two points in $\mathscr{X}$ cannot have the same sign sequence, for then their spheres of influence would not both contain O. There are exactly 3^d distinct sign sequences of length d. Therefore $n \leqslant 3^d$. $\square$

A $\{G_1, \ldots, G_m\}$-*factor* of a graph G is a spanning subgraph of G in which each connected component is isomorphic to a graph in the set $\{G_1, \ldots, G_m\}$. In [8] it is shown that *for any metric space M every M-*CSIG *must possess a $\{K(2), K(3)\}$-factor.* Thus it will not be possible to define the "CSIG-dimension" of a graph in general. Note that if a tree is an M_∞^d-CSIG, then it must have a $\{K(2)\}$-factor, i.e., a perfect matching. A special case of Theorem 13 in [8] establishes the converse for $d \geqslant 2$ and thereby characterizes trees that are M_∞^d-CSIGs:

Proposition 16. *For $d \geqslant 2$ a tree is an M_∞^d-*CSIG *if and only if it has a perfect matching.*

9. Open problems

We end this paper by formulating some problems and a conjecture about M_∞^d-SIGs and the SIG-dimension. We begin with a fundamental problem.

Problem 17. *Characterize the set of graphs with fixed* SIG-*dimension d in some reasonable manner* ($d \geqslant 2$).

The following "star-factor" theorem is a consequence of general results (Theorems 3 and 12) of Michael and Quint [8] and sheds some light on Problem 17.

Proposition 18. *Every M_∞^d-*SIG *has a $\{K(1,1), \ldots, K(1,2^d)\}$-factor.*

Our next theorem summarizes what is known about Problem 17 for $d = 2$.

Theorem 19. *Let G be an M_∞^2-*SIG. *Then*
(a) *G has a $\{K(1,1), \ldots, K(1,4)\}$-factor*;
(b) *every non-planar induced subgraph of G contains $K(3)$ as a subgraph*;
(c) *there exists a 2-coloring of the edges of G with no monochromatic $K(3)$.*

Proof. Conclusion (a) is the case $d = 2$ of Proposition 18. Conclusion (b) is a special case of Theorem 5 in [8]. Conclusion (c) is the special case $d = 2$, $n = 3$ of Theorem 2. $\square$

Let $M_1^d = (\mathbf{R}^d, \rho_1)$ denote the metric space induced by the L_1-metric on $\mathbf{R}^d$. Now if $\mathscr{X}'$ is the set of points in the plane obtained by rotating the set $\mathscr{X}$ through an angle of 45°, then the SIGs M_∞^2-SIG($\mathscr{X}$) and M_1^2-SIG($\mathscr{X}'$) are isomorphic. Thus the *sets* of M_∞^2-SIGs and M_1^2-SIGs are identical. This relationship between the sets of L_1- and L_∞-SIGs fails for higher dimensions.

Trees are among the simplest graphs, yet we know of no general formula for the SIG-dimension of an arbitrary tree.

Problem 20. *Find a formula for the* SIG-*dimension of a tree* (*say*, *in terms of its degree sequence and other graphical parameters*).

The bounds in Theorem 9 are our best general results concerning the SIG-dimensions of trees. Note that Proposition 18 does not provide a characterization of trees with SIG-dimension d. For instance, the unique tree T_d on $2^d + 3$ vertices with degree sequence $(1, \ldots, 1, 2, 2^d + 1)$ has a $\{K(1,1), \ldots, K(1, 2^d)\}$-factor, but one may readily show that $\mathrm{sig}(T_d) \geqslant d + 1$.

We remark that the Euclidean planar analogue of Proposition 18 asserts that if G is a Euclidean planar SIG, then G has a $\{K(1,1), K(1,2)\}$-factor. (See [8].) Jacobson, Lipman, and McMorris [4] established the converse for trees. Thus the trees that are Euclidean planar SIGs have been characterized, but the analogous problem for M_∞^2-SIGs remains open.

Problem 21. *Characterize the trees with* SIG-*dimension* 2.

We seek an extension of formula (4) that treats the SIG-dimensions of complete multipartite graphs in which several partite sets have cardinality 1.

Problem 22. *Find the* SIG-*dimension of all complete multipartite graphs.*

Progress on Problem 22 has been reported by Boyer et al. [1].

Let Q_m denote the *cube graph* on 2^m vertices. The vertices of Q_m correspond to strings of 0's and 1's of length m; two vertices are joined by an edge provided the corresponding strings differ in exactly one coordinate.

Problem 23. *Find the* SIG-*dimension of the cube graph* Q_m.

Clearly, $\mathrm{sig}(Q_1) = 1$, and it is not difficult to see that $\mathrm{sig}(Q_2) = 2$. Moreover, the set $\{(\pm 1, 0), (0, \pm 1), (\pm 3, 0), (0, \pm 3)\}$ in $\mathbf{R}^2$ realizes Q_3, and thus $\mathrm{sig}(Q_3) = 2$. For $m \geqslant 4$ the graph Q_m is non-planar and contains no $K(3)$, and hence $\mathrm{sig}(Q_m) \geqslant 3$ by Theorem 19(b). Also, Corollary 6 implies that $\mathrm{sig}(Q_m) \geqslant \lceil \log_2(m) \rceil$. The authors have an inductive construction that shows that $\mathrm{sig}(Q_m) \leqslant m$.

Let us turn to a fundamental algorithmic problem concerning SIGs.

Problem 24. *Let G be a graph without isolated vertices, and let d be a positive integer. What is the computational complexity of testing* $\mathrm{sig}(G) = d$?

Presumably a solution to Problem 24 will allow one to construct a set of points in $\mathbf{R}^d$ that realizes G, where $d = \mathrm{sig}(G)$.

Let G be a connected graph with n vertices ($n \geqslant 2$). Although we know of no finite algorithm to compute the SIG-dimension of G, we may compute the unitary SIG-dimension $\mathrm{sig}^*(G)$ as follows: For $d = 1, \ldots, n$ we simply examine all induced subgraphs of Z^d_∞ with n vertices; it suffices to restrict attention to the subgraphs of Z^d_∞ with all components in $\{1, \ldots, n\}$. Lemma 12 assures us that the smallest d for which G is an induced subgraph of Z^d_∞ is the unitary SIG-dimension of G.

Finally, we ask for the maximum edge density of SIGs and CSIGs in M^d_∞. Define the *edge density constants*

$$C_d = \sup\{|E|/|V| : G = (V, E) \text{ is an } M^d_\infty\text{-SIG}\},$$

$$\bar{C}_d = \sup\{|E|/|V| : G = (V, E) \text{ is an } M^d_\infty\text{-CSIG}\}.$$

Problem 25. *Determine the values of the edge density constants C_d and $\bar{C}_d$.*

It is easy to see that $C_1 = 1$ and $\bar{C}_1 = 2$. The following general inequalities of Soss [10] imply that $C_2 = 6$ and also disprove a conjecture in [7].

Proposition 26. *The edge density constant for* SIGs *under the L_∞-metric satisfies*

$$(4^{d+1} - 3d - 4)/9 \leqslant C_d \leqslant 2^{2d-1} - 2^{d-1}.$$

The edge density constant $\bar{C}_d$ is known to satisfy

$$(4^{d+1} - 3d - 4)/9 \leqslant C_d \leqslant \bar{C}_d \leqslant 5^d - 1.5. \tag{6}$$

The leftmost inequality in (6) is from Proposition 26, and the middle inequality is clear. The rightmost inequality in (6) is valid for the edge density constant associated with *any* metric induced by a norm on $\mathbf{R}^d$, not only the L_∞-metric; see Theorem 3 of [6].

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